THE CHARACTER OF BOUNDARY LAYERS AND THEIR OVERLAPPING IN THE MAGNETOHYDRODYNAMIC FLOW OF A CONDUCTING FLUID IN A CHANNEL OF RECTANGULAR CROSS SECTION

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G.A. GRINBERG and O. M. KONTOROVICH
(Leningrad)

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Green's function is used as a basis for considering the flow of a conducting fluid in a pipe of rectangular cross section with nonconductive walls under a transverse magnetic field (the Shercliff problem). The distributions of the velocity and induced magnetic field next to the walls which are parallel to the external magnetic field and near the corners of the pipe cross section are investigated in detail, especially for large Hartmann numbers. The results obtained are illustrated graphically.

1. The steady-state flow of a viscous conducting fluid in a rectangular pipe under a transverse magnetic field is investigated in several papers [1 to 5]. We know [1] that if the external magnetic field H° is homogeneous and if the velocity field and the induced electric and magnetic fields are independent of the coordinate z measured in the direction of the pipe axis, then there exists a solution of the equations of steady motion of the conducting viscous incompressible fluid in the pipe for which the velocity v and the induced magnetic field H are parallel to the z-axis and satisfy Eqs.

$$\Delta H + \frac{4\pi\mu\sigma H^\circ}{c^2}\frac{\partial v}{\partial x} = 0 \qquad (0 < x < l) \tag{1.1}$$

$$\Delta v + \frac{\mu H^\circ}{4\pi\eta}\frac{\partial H}{\partial x} = -\frac{P^\circ}{\eta} \qquad (0 < y < d) \tag{1.2}$$

The x-axis lies in the direction of the field H°; σ, μ, η are the conductivity, magnetic permeability, and coefficient of viscosity of the fluid, respectively; c is the velocity of light; $\partial p/\partial z = P^\circ$ is the pressure gradient which is assumed to be constant over the pipe cross section.

The boundary conditions at the channel walls, which we assume to be nonconductive, reduce to the vanishing of both v and H at the pipe contour s.

This solution of this problem is given in [1] in the form of infinite trigonometric series whose convergence deteriorates with increasing Hartmann numbers M. Shercliff showed that for large M his solution is practically identical to the ordinary solution of the one-dimensional Hartmann problem except in the regions adjacent to the channel walls $y = 0$ and $y = d$. The same author also obtained a solution of an approximate equation applicable for large numbers M near the walls parallel to the external magnetic field except for points in the immediate vicinity of the corners. He assumed the second derivatives with respect to x in Eqs. (1.1) and (1.2) to be small as compared with the other terms and therefore negligible. The equations obtained in this way were solved exactly by way of an elegant substitution.

Another method of solving the problem is suggested in [2] and [3]. It involves the use of the Green function in an Eq. of the form

$$\Delta u - m^2 u = 0, \qquad m = (\mu H^\circ/2c)\sqrt{\sigma/\eta}$$

This method makes it possible to obtain solutions directly and in a form convenient for practical applications for large Hartmann numbers and at all points of the channel cross section.

We shall use the method here to obtain the velocity and magnetic field distributions in the boundary layer parallel to the magnetic field H° and in the neighborhood of the corner point of the contour.

2. In accordance with the usual procedure, we begin by introducing the functions

$$w_{\pm} = v + qmy^2 \pm \alpha H$$

$$\alpha = \frac{c}{4\pi\sqrt{\sigma\eta}}, \qquad q = \frac{P^{\circ}}{2\eta m} \tag{2.1}$$

for which we obtain Eqs.

$$\Delta w_{\pm} \pm 2m\partial w_{\pm}/\partial x = 0 \tag{2.2}$$

with the boundary conditions

$$w_{\pm}|_s = qm\eta^2(s) \tag{2.3}$$

Further, setting

$$w_{\pm} = e^{\mp mx}\varphi_{\pm} \tag{2.4}$$

we obtain for the functions $\varphi_{\pm}$ the following Eq.:

$$\Delta\varphi_{\pm} - m^2\varphi_{\pm} = 0 \tag{2.5}$$

with the boundary conditions

$$\varphi_{\pm}|_s = e^{\pm m\xi(s)} qm\eta^2(s) \tag{2.6}$$

Then, in accordance with [3], we have

$$v = -qmy^2 + qm\int_{(s)} \eta^2\, chm(x-\xi)\frac{\partial G}{\partial n}ds \tag{2.7}$$

$$\alpha H = qm\int_{(s)} \eta^2 shm(\xi - x)\frac{\partial G}{\partial n}ds \tag{2.8}$$

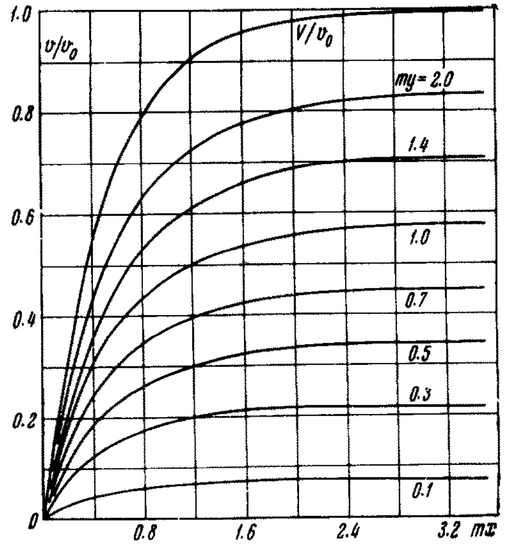

Fig. 1

where $G(x, y, \xi, \eta)$ is the corresponding Green function, (x, y) is a fixed point of the domain bounded by the contour s, (ξ, η) is a point on this contour, and n is the exterior normal.

3. Assuming that $md \gg 1$ and $ml \gg 1$, we can be content in investigating the flow near the channel wall $y = 0$ with a few terms of the Green function series, since these terms diminish rapidly with a large Hartmann number. Thus, we set

$$\begin{aligned} G = -{}^1\!/_2\pi^{-1}\{&K_0[m\sqrt{(x-\xi)^2+(y-\eta)^2}] - K_0[m\sqrt{(x-\xi)^2+(y+\eta)^2}] - \\ &- K_0[m\sqrt{(x+\xi)^2+(y-\eta)^2}] + K_0[m\sqrt{(x+\xi)^2+(y+\eta)^2}] - \\ &- K_0[m\sqrt{(2l-x-\xi)^2+(y-\eta)^2}] + K_0[m\sqrt{(2l-x-\xi)^2+(y+\eta)^2}] + \\ &+ K_0[m\sqrt{(2l+x-\xi)^2+(y-\eta)^2}] - K_0[m\sqrt{(2l+x-\xi)^2+(y+\eta)^2}] \end{aligned} \tag{3.1}$$

where the terms of orders e^{-md} and e^{-2ml} have been omitted. The velocity for $0 < x < l/2$ is then given by Formula

$$\begin{aligned} v = -qmy^2 + \frac{qm^2x}{\pi}chmxF(x, y) + \frac{qm^2(l-x)}{\pi}chm(l-x)F[(l-x); y] - \\ - \frac{qm^2(l+x)}{\pi}chm(l-x)F[(l+x); y] \end{aligned} \tag{3.2}$$

$$F(a; y) = \int_0^d \eta^2\left\{\frac{K_1[m\sqrt{a^2+(y-\eta)^2}]}{\sqrt{a^2+(y-\eta)^2}} - \frac{K_1[m\sqrt{a^2+(y+\eta)^2}]}{\sqrt{a^2+(y+\eta)^2}}\right\}d\eta \tag{3.3}$$

Since $y \ll d$, it follows that $m(d-y) \gg 1$, $m(d+y) \gg 1$, so that by a simple substitution of variables we can reduce Formula (3.3) to the more convenient form

$$F(a;\,y) = \frac{4y}{m} K_0\left[m\sqrt{a^2+y^2}\right] + 2\int_0^y (y^2+t^2)\frac{K_1(m\sqrt{a^2+t^2})}{\sqrt{a^2+t^2}}\,dt \tag{3.4}$$

Similarly, we can derive Formula

$$\alpha H = -\frac{qm^2x}{\pi}\,\mathrm{sh}\,mx\,F(x,\,y) + \frac{qm^2(l-x)}{\pi}\,\mathrm{sh}\,m(l-x)\,F\left[(l-x),\,y\right] - \\ -\frac{qm^2(l+x)}{\pi}\,\mathrm{sh}\,m(l-x)\,F\left[(l+x),\,y\right] \tag{3.5}$$

Figs. 1 to 3 show curves computed using Formulas (3.2) and (3.5) for the case $ml = 7$ in the range $0 \leqslant x \leqslant l/2$, where $c = my$, $v_0 = ml/2$, and V is the velocity in the corresponding Hartmann problem.

4. In conclusion let us compare this solution with the approximate solution obtained by Shercliff in [1] for the boundary layer at points near the wall $y = 0$ and at a sufficient distance from the corner, i.e. for

$$\frac{m^2y^2}{mx} \ll 1, \qquad m(l-x) \gg 1 \tag{4.1}$$

Let us consider the function $v_- = v -$

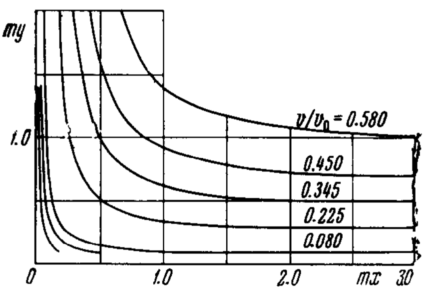

Fig. 2

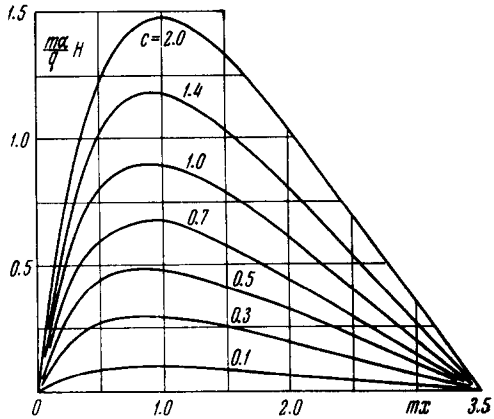

Fig. 3

$-\alpha H$ introduced by Shercliff, for which in accordance with (3.2) and (3.5) we obtain Expression

$$v_- = v - \alpha H = -qmy^2 + \frac{qm^2x}{\pi}e^{mx}F(x;\,y) + \\ + \frac{qm^2(l-x)}{\pi}e^{-m(l-x)}F\left[(l-x);\,y\right] - \frac{qm^2(l+x)}{\pi}e^{-m(l-x)}F\left[(l+x);\,y\right] \tag{4.2}$$

Since $m(l-x) \gg 1$, we can neglect terms of the order $e^{-2m(l-x)}$ to obtain

$$v_- = -qmy^2 + \frac{qm^2x}{\pi}F(x;\,y) \tag{4.3}$$

i.e.

$$v_- = -qmy^2 + \frac{qm^2x}{\pi}\left\{\frac{4y}{m}K_0(m\sqrt{x^2+y^2}) + 2\int_0^y (y^2+t^2)\frac{K_1(m\sqrt{x^2+t^2})}{\sqrt{x^2+t^2}}\,dt\right\} \tag{4.4}$$

Making use of asymptotic representations of the Macdonald function and taking account of conditions (4.1), we obtain the following Expression for F:

$$F(x,\,y) = \frac{4y}{m}\sqrt{\frac{\pi}{2mx}}\,e^{-mx}\left(1+\frac{m^2y^2}{6mx}\right) \tag{4.5}$$

Substituting (4.5) into (4.3), we have

$$v_- = -qmy^2 + \frac{qm^2x}{\pi}\sqrt{\frac{\pi}{2mx}}\,\frac{4y}{m}\left(1+\frac{m^2y^2}{6mx}\right) \tag{4.6}$$

Following Shercliff's procedure and introducing the new variable

$$u = y\sqrt{2m/x} \tag{4.7}$$

we obtain

$$v_- = qx\frac{2}{\sqrt{\pi}}u\left(1-\frac{u^2}{12}\right)-\frac{qx}{2}u^2 = qx\frac{2}{\sqrt{\pi}}\left(1-\frac{\sqrt{\pi}}{4}u+\frac{u^2}{12}\right)u =$$
$$= 1.13\,qx\,(u - 0.443u^2 + 0.083u^3) \tag{4.8}$$

In [1] the function v_- is given by Formula

$$v_- = qx\left\{1-\left(1+\frac{u^2}{2}\right)\left[1-\left(\int_0^u \exp\frac{-u^2}{4}\frac{du}{(2+u^2)^2}\right)\left(\int_0^\infty \exp\frac{-u^2}{4}\frac{du}{(2+u^2)}\right)^{-1}\right]\right\} \tag{4.9}$$

$$\int_0^\infty \exp\frac{-u^2}{4}\frac{du}{(2+u^2)^2} = 0.23 \tag{4.10}$$

Then, to within terms of the order u^3 Formula (4.9) yields

$$v_- = qx\left\{1-\left(1+\frac{u^2}{2}\right)\left[1-\frac{1}{0.92}\left(u-\frac{5}{12}u^3\right)\right]\right\} = 1.08\,qx\,(u-0.46u^2+0.083u^3) \tag{4.11}$$

Comparison with (4.8) shows that in the region considered Formula (4.11) is close to the exact one.

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